例2：计算 $\sum_{k=1}^{n}\cos k\alpha$ $\sum_{k=1}^{n}\sin k\alpha$.

解 设 $z=\cos\alpha+i\sin\alpha$. $z_k=z^k$ 则 $z_k=\cos k\alpha+i\sin k\alpha$

故 $\sum_{k=1}^{n}\cos k\alpha+i\sum_{k=1}^{n}\sin k\alpha=\sum_{k=1}^{n}(\cos k\alpha+i\sin k\alpha)$

$$=\sum_{k=1}^{n}z^k=\frac{(1-z^n)z}{1-z}=\frac{(\cos\alpha+i\sin\alpha)(1-\cos n\alpha-i\sin n\alpha)}{1-\cos\alpha-i\sin\alpha}$$

$$=\frac{(\cos\alpha+i\sin\alpha)\left(-2\sin\frac{n\alpha}{2}\left(\cos\frac{\pi+n\alpha}{2}+i\sin\frac{n+n\alpha}{2}\right)\right)}{-2\sin\frac{\alpha}{2}\left(\cos\frac{\pi+\alpha}{2}+i\sin\frac{\pi+\alpha}{2}\right)}$$

$$=\frac{\sin\frac{n\alpha}{2}\cos\frac{n+1}{2}\alpha}{\sin\frac{\alpha}{2}}+i\,\frac{\sin\frac{n\alpha}{2}\sin\frac{n+1}{2}\alpha}{\sin\frac{\alpha}{2}}$$

对应地，有 $\sum_{k=1}^{n}\cos k\alpha=\frac{\sin\frac{n\alpha}{2}\cos\frac{n+1}{2}\alpha}{\sin\frac{\alpha}{2}}$

$$\sum_{k=1}^{n}\sin k\alpha=\frac{\sin\frac{n\alpha}{2}\sin\frac{n+1}{2}\alpha}{\sin\frac{\alpha}{2}}$$

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4.6 三角函数

5. 设 n 是正整数，$\theta \in \mathbb{R}$. 证明:

$$\sum_{k=0}^{n} \binom{n}{k} \cos k\theta = 2^n \cos^n \frac{\theta}{2} \cos \frac{n\theta}{2}$$

证明: 注意到 $(e^{i\theta}+1)^n = \sum_{k=0}^{n} \binom{n}{k} e^{ik\theta}$

又 $\sum_{k=0}^{n} \binom{n}{k} e^{ik\theta} = \sum_{k=0}^{n} \binom{n}{k} \cos k\theta + i \sum_{k=0}^{n} \binom{n}{k} \sin k\theta$

而 $e^{i\theta}+1 = \cos\theta + 1 + i\sin\theta = 2\cos^2\frac{\theta}{2} + 2i\cos\frac{\theta}{2}\sin\frac{\theta}{2} = 2\cos\frac{\theta}{2}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})$

即 $(e^{i\theta}+1)^n = 2^n \cos^n\frac{\theta}{2}(\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2})$ (De-Moivre公式)

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